Best Fidelity Conditions for Three Party Quantum Teleportation

Sangjin Sim, Jeongwan Jin and Younghun Kwon*

Department of Physics, Hanyang University,

Ansan, Kyunggi-Do, 425-791, South Korea

(Dated: February 1, 2008)

Recently Acin et al. classified three qubit states and showed that there would be various types of three qubit entangled states. Using the entangled three qubit states classified by Acin et al. we consider the quantum teleportation among three parties. And we find the best fidelity conditions for the quantum teleportation among three parties.

PACS numbers: 03.67.Lx

Quantum teleportation is an efficient way to transmit a quantum information when classical communications(CC) and local operations(LO) are allowed to parties already sharing an entangled state. The quantum teleportation between two parties was proposed by Bennet et al[1]. In quantum teleportation of two parties, Alice(sender) and Bob(receiver) share a maximally entangled state. Alice attaches the information state to shared entangled state and performs Bell basis mesurement. And she sends her result to Bob. According to Alice's Bell basis measurement, Bob applies the corresponding unitary operation on his single qubit and obtains the original information state with certainty.

Quantum teleportation of three parties using a three qubit entangled state was introduced by Karsson et al[2]. The main difference between quantum teleportation of the two parties and that of three parties is the concept of cosender. A sender first performs Bell basis measurement on his(or her) two qubits,(one is the information qubit and the other is the qubit entangled to other parties) and sends the measurement result to the co-sender and the receiver.The co-sender performs single qubit measurement according to the sender's measurement result and sends the measurement result to the receiver.Given the protocol provided in the secret when three parties are separated, the receiver performs local unitary operations according to the measurement results. Then the receiver can recover the information state with a probability.

---

*Electronic address: yyhkwon@hanyang.ac.kr

Recently Acin et al. showed that there are 7 different entangled states in three qubit states[4]. In this paper, we consider quantum teleportation in three parties with those entangled states.In fact Yeo considered the quantum teleportation among three parties, using GHZ state and W state[3]. So we will consider the quantum teleportation among three parties, using the other entangled states except GHZ state and W state. For each case, we will provide the fidelity, the best fidelity condition and teleportation protocols.

This paper is organized as follows. In the section I, we first review the three party quantum teleportation with GHZ state and W state. In the section II, we consider quantum teleportation in three parties sharing different entangled states based on Acin et al.'s classification of three qubit states. And the roles of parties, sender, co-sender, and receiver, are determined. Also we give the best fidelity conditions for each case. In the section III, we summarize and discuss our results.

I. QUANTUM TELEPORTATION IN THREE PARTIES WITH SYMMETRIC THREE-QUBIT STATES

Let us review quantum teleportation among three parties. Quantum teleportation in three parties sharing a three-qubit entangled state consists of three steps:

1)First, three parties shares a three qubit entangled state. A sender performs Bell basis measurement on his(or her) two qubits,(one is the information qubit and the other the qubit entangled to other parties) and sends the measurement result j to the co-sender and the receiver. The Bell basis measurement makes use of the following projection operators : $|\Phi^+\rangle\langle\Phi^+|$ for $j=1$, $|\Phi^-\rangle\langle\Phi^-|$ for $j=2$, $|\Psi^+\rangle\langle\Psi^+|$ for $j=3$, and $|\Psi^-\rangle\langle\Psi^-|$ for $j=4$, where

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \quad (1)$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \quad (2)$$

$$(3)$$

2)The co-sender performs single qubit measurement, according to the sender's measurement result j, and sends the measurement result k to the receiver. The single qubit measurement applies the following pojections : $|\mu^+\rangle\langle\mu^+|$ for $k=1$ and $|\mu^-\rangle\langle\mu^-|$ for $k=2$, where

$$|\mu^{+}\rangle \; = \; \sin\nu|0\rangle + e^{i\kappa}\cos\nu|1\rangle \tag{4}$$

$$|\mu^{-}\rangle \; = \; \cos\nu|0\rangle - e^{i\kappa}\sin\nu|1\rangle \tag{5}$$

$$\tag{6}$$

3)Given the protocol provided in the secret when three parties are separated, the receiver performs local unitary operations according to the measurement results j and k. Then the party recovers the information state with a probability.

If $|\tau\rangle$ denotes the receiver's reconstructed state, then the success rate is measured by the fidelity of the original information state, $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$, and the $|\tau\rangle$, which is

$$\langle F\rangle = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \sum_{j,k} |\langle\psi|\tau\rangle|^2 \tag{7}$$

When the GHZ state is shared in three parties, the fidelity is shown as $\langle F_{GHZ}\rangle = \frac{2}{3} + \frac{1}{3}\sin 2\nu$ given the protocol in table I.

	$j=1$	$j=2$	$j=3$	$j=4$
$k=1$	I	σ_z	σ_x	σ_y
$k=2$	σ_z	I	σ_y	σ_x

TABLE I: The protocol of the quantum teleportation in three parties when the GHZ states is applied

When the W state is shared in three parties, the fidelity is shown as $\langle F_W\rangle = 7/9$ given the protocol in table II.

	$j=1$	$j=2$	$j=3$	$j=4$
$k=1$	σ_x	σ_y	I	σ_z
$k=2$	σ_x	σ_y	I	σ_z

TABLE II: The protocol of the quantum teleportation in three parties when the W states is applied

We here note that $\langle F_W \rangle > \langle F_{GHZ} \rangle$ in average. However if $sin 2\nu$ is greater than $\frac{1}{3}$, $F_{GHZ} > F_W$. And the best fidelity condition for F_{GHZ} is $\nu = \frac{\pi}{4} + m\pi$. Here the best fidelity condition means that if the co-sender Bob can perform his single qubit measurement, according to the sender's measurement result j, using the following pojections : $|\mu^+\rangle\langle\mu^+|$ for $k=1$ and $|\mu^-\rangle\langle\mu^-|$ for $k=2$, where

$$|\mu^+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \tag{8}$$

$$|\mu^-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \tag{9}$$

(10)

then the fidelity produces the best result.

II. Quantum teleportation with asymmetric states

The classification of three-qubit state by Acin et al. is as follows;

Type 1 (Product states)

Type 2a (Biseparable states)

$$
\begin{aligned}
|2aI\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |100\rangle + |101\rangle) \\
|2aII\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |100\rangle + |110\rangle)
\end{aligned}
\tag{11}
$$

Type 2b (GHZ state)

$$
|2b\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)
\tag{12}
$$

Type 3a (Tri-Bell state)

$$
|3a\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |101\rangle + |110\rangle)
\tag{13}
$$

Type 3b (Extended GHZ states)

$$
\begin{aligned}
|3bI\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |110\rangle + |111\rangle) \\
|3bII\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |100\rangle + |111\rangle) \\
|3bIII\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |101\rangle + |111\rangle)
\end{aligned}
\tag{14}
$$

Type 4a

$$
|4a\rangle = \frac{1}{\sqrt{4}}(|000\rangle + |100\rangle + |101\rangle + |110\rangle)
\tag{15}
$$

Type 4b

$$\begin{aligned} |4bI\rangle &= \frac{1}{\sqrt{4}}(|000\rangle + |100\rangle + |110\rangle + |111\rangle) \\ |4bII\rangle &= \frac{1}{\sqrt{4}}(|000\rangle + |100\rangle + |101\rangle + |111\rangle) \end{aligned} \tag{16}$$

Type 4c

$$|4c\rangle = \frac{1}{\sqrt{4}}(|000\rangle + |101\rangle + |110\rangle + |111\rangle) \tag{17}$$

Type 5 (Real states)

$$|5\rangle = \frac{1}{\sqrt{5}}(|000\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \tag{18}$$

Note that the tri-Bell state is equivalent to the W state. Therefore, we need to consider only type3-5 states.

We now show all schemes of quantum teleportation in three parties sharing one of the these types. Note the reference of protocols. The two W protocols are equivalent with respect to a permutation of parties.

	$j=1$	$j=2$	$j=3$	$j=4$
$k=1$	I	σ_z	σ_x	σ_y
$k=2$	σ_z	I	σ_y	σ_x

TABLE III: GHZ protocol

	$j=1$	$j=2$	$j=3$	$j=4$
$k=1$	I	σ_z	σ_x	σ_y
$k=2$	I	σ_z	σ_x	σ_y

TABLE IV: W protocol I

	$j=1$	$j=2$	$j=3$	$j=4$
$k=1$	σ_x	σ_y	I	σ_z
$k=2$	σ_x	σ_y	I	σ_z

TABLE V: W protocol II

A. Quantum teleportation in three parties sharing the extended GHZ state

An extended GHZ state is transformed to another extended GHZ state under permutations of parties. Thus, it is sufficient to consider the case that Alice, Bob and Cindy share the following state

$$|\psi\rangle \;=\; \frac{1}{\sqrt{3}}(|0_A0_B0_C\rangle + |0_A1_B1_C\rangle + |1_A1_B1_C\rangle) \tag{19}$$

Suppose that they want to teleport the information state $\cos(\theta/2)|0\rangle + e^{i\kappa}\sin(\theta/2)|1\rangle$. We know that Bob and Cindy are symmetric to a permutation of them. There are four choices in determining their roles of quantum teleportation. We will use '$\rightarrow$' to mean that a party sends the measurement result to another one via CC(one-way) and '$\leftrightarrow$' to mean that both $\rightarrow$ and $\leftarrow$ are possible via CC(two-way).

1. Alice(sender) $\rightarrow$ Bob $\leftrightarrow$ Cindy

The fidelity is $\frac{5}{9} + \frac{2}{9}\cos\kappa\sin 2\nu$ and the protocol is of GHZ. The best fidelity condition is $\kappa = 2n\pi, \nu = \frac{\pi}{4} + m\pi$

2. Bob(sender) $\rightarrow$ Alice(co-sender) $\rightarrow$ Cindy(receiver)

The fidelity is $\frac{8}{9}$, and the protocol is of W.

3. Bob(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Alice(receiver)

The fidelity is $\frac{5}{9} + \frac{2}{9}\cos\kappa\sin 2\nu$, and the protocol is of GHZ. The best fidelity condition is $\kappa = 2n\pi, \nu = \frac{\pi}{4} + m\pi$

B. Quantum teleportation in three parties sharing the type4a state

Suppose that Alice, Bob and Cindy shared the following state

$$
\begin{aligned}
|\psi\rangle \; &= \; \frac{1}{\sqrt{4}}(|0_A 0_B 0_C\rangle + |1_A 0_B 0_C\rangle + |1_A 0_B 1_C\rangle \\
&\quad +|1_A 1_B 0_C\rangle)
\end{aligned}
\tag{20}
$$

Since Bob and Cindy are symmetric parties, there are four cases as follows,

1. Alice(sender) $\rightarrow$ Bob $\leftrightarrow$ Cindy

The fidelity is $\frac{2}{3}$ and the protocol is of W.

2. Bob(sender) $\rightarrow$ Alice(co-sender) $\rightarrow$ Cindy(receiver)

The fidelity is $\frac{2}{3}$ and the protocol is of the second W.

3. Bob(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Alice(receiver)

The fidelity is $\frac{2}{3}$ and the protocol is of W.

C. Quantum teleportation in three parties sharing the type4b state

Suppose that Alice, Bob and Cindy shared the following state

$$
\begin{aligned}
|\psi\rangle \;=\; & \frac{1}{\sqrt{4}}(|0_A 0_B 0_C\rangle + |1_A 0_B 0_C\rangle + |1_A 1_B 0_C\rangle \\
& + |1_A 1_B 1_C\rangle)
\end{aligned}
\tag{21}
$$

Since there are no symmetric parties, there are six cases as follows,

1. Alice(sender) $\rightarrow$ Bob(co-sender) $\leftrightarrow$ Cindy(receiver)

The fidelity is $\frac{1}{2} + \frac{1}{6}\cos\kappa\sin 2\nu$ and the protocol is of GHZ. The best fidelity condition is $\kappa = 2n\pi, \nu = \frac{\pi}{4} + m\pi$

2. Alice(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Bob(receiver)

The fidelity is $\frac{3}{4}$ and the protocol is of W.

3. Bob(sender) $\rightarrow$ Alice $\rightarrow$ Cindy

The fidelity is $\frac{7}{12} + \frac{1}{6}\cos 2\nu + \frac{1}{6}\cos\kappa\sin 2\nu$ and the protocol is of GHZ. The best fidelity condition is $\kappa = 2n\pi, \nu = \frac{\pi}{8} + m\pi$

4. Bob(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Alice(receiver)

The fidelity is $\frac{3}{4}$ and the protocol is of W.

5. Cindy(sender) $\rightarrow$ Alice(co-sender) $\rightarrow$ Bob(receiver)

The fidelity is $\frac{7}{12} + \frac{1}{6}\cos 2\nu + \frac{1}{6}\cos\kappa\sin 2\nu$ and the protocol is of GHZ. The best fidelity condition is $\kappa = 2n\pi, \nu = \frac{\pi}{8} + m\pi$

6. Cindy(sender) $\rightarrow$ Bob(co-sender) $\rightarrow$ Alice(receiver)

The fidelity is $\frac{1}{2}+\frac{1}{6}\cos\kappa\sin 2\nu$ and the protocol is of GHZ. The best fidelity condition is $\kappa = 2n\pi, \nu = \frac{\pi}{4} + m\pi$

D. Quantum teleportation in three parties sharing the type4c state

Suppose that Alice, Bob and Cindy shared the following state

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{4}}(|0_A 0_B 0_C\rangle + |1_A 0_B 1_C\rangle + |1_A 1_B 0_C\rangle \\ &\quad + |1_A 1_B 1_C\rangle) \end{aligned} \tag{22}$$

Since Bob and Cindy are symmetric parties, there are four cases as follows,

1. Alice(sender) $\rightarrow$ Bob $\leftrightarrow$ Cindy

The fidelity is $\frac{3}{4}$ and the protocol is of W.

2. Bob(sender) $\rightarrow$ Alice(co-sender) $\rightarrow$ Cindy(receiver)

The fidelity is $\frac{1}{2}+\frac{1}{6}\cos\kappa\sin 2\nu$ and the protocol is of the second GHZ. The best fidelity condition is $\kappa = 2n\pi, \nu = \frac{\pi}{4} + m\pi$

3. Bob(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Alice(receiver)

The fidelity is $\frac{3}{4}$ and the protocol is of W.

E. Quantum teleportation in three parties sharing the type5 state

Suppose that Alice, Bob and Cindy shared the following state

$$|\psi\rangle = \frac{1}{\sqrt{4}}(|0_A 0_B 0_C\rangle + |1_A 0_B 0_C\rangle + |1_A 0_B 1_C\rangle + |1_A 1_B 0_C\rangle + |1_A 1_B 1_C\rangle) \tag{23}$$

Since Bob and Cindy are symmetric parties, there are four cases as follows,

1. Alice(sender) $\rightarrow$ Bob $\leftrightarrow$ Cindy

The fidelity is $\frac{2}{3}$ and the protocol is of W.

2. Bob(sender) $\rightarrow$ Alice(co-sender) $\rightarrow$ Cindy(receiver)

The fidelity is $\frac{8}{15} + \frac{2}{15}\cos 2\nu + \frac{2}{15}\cos\kappa \sin 2\nu$ and the protocol is of GHZ. The best fidelity condition is $\kappa = 2n\pi, \nu = \frac{\pi}{8} + m\pi$

3. Bob(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Alice(receiver)

The fidelity is $\frac{2}{3}$ and the protocol is of W.

All scheme of quantum teleportatopn among three parties are shown in Table VI.

We here note that there are only two protocols W and GHZ in tables[III] and [IV]. This is due to the different entanglement structure of W and GHZ states. In other words, W state cannot be transformed to GHZ state with a probability. This implies that protocols can classify quantum states like stochastic LOCC. That is, we classify the

	role	*fidelity*	*protocol*	*bestfidelitycondition*
extendedGHZ	Alice(sender) $\rightarrow$ Bob $\leftrightarrow$ Cindy	GHZ	$\frac{5}{9}+\frac{2}{9}\cos\kappa\sin 2\nu$	$\kappa=2n\pi,\nu=\frac{\pi}{4}+m\pi$
	Bob(sender) $\rightarrow$ Alice(co-sender) $\rightarrow$ Cindy(receiver)	W	$\frac{8}{9}$	
	Bob(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Alice(receiver)	GHZ	$\frac{5}{9}+\frac{2}{9}\cos\kappa\sin 2\nu$	$\kappa=2n\pi,\nu=\frac{\pi}{4}+m\pi$
type4a	Alice(sender) $\rightarrow$ Bob $\leftrightarrow$ Cindy	W	$\frac{2}{3}$	
	Bob(sender) $\rightarrow$ Alice(co-sender) $\rightarrow$ Cindy(receiver)	W	$\frac{2}{3}$	
	Bob(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Alice(receiver)	W	$\frac{2}{3}$	
type4b	Alice(sender) $\rightarrow$ Bob(co-sender) $\leftrightarrow$ Cindy(receiver)	GHZ	$\frac{1}{2}+\frac{1}{6}\cos\kappa\sin 2\nu$	$\kappa=2n\pi,\nu=\frac{\pi}{4}+m\pi$
	Alice(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Bob(receiver)	W	$\frac{3}{4}$	
	Bob(sender) $\rightarrow$ Alice $\rightarrow$ Cindy	GHZ	$\frac{7}{12}+\frac{1}{6}\cos 2\nu+\frac{1}{6}\cos\kappa\sin 2\nu$	$\kappa=2n\pi,\nu=\frac{\pi}{8}+m\pi$
	Bob(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Alice(receiver)	W	$\frac{3}{4}$	
	Cindy(sender) $\rightarrow$ Alice(co-sender) $\rightarrow$ Bob(receiver)	GHZ	$\frac{7}{12}+\frac{1}{6}\cos 2\nu+\frac{1}{6}\cos\kappa\sin 2\nu$	$\kappa=2n\pi,\nu=\frac{\pi}{8}+m\pi$
	Cindy(sender) $\rightarrow$ Bob(co-sender) $\rightarrow$ Alice(receiver)	GHZ	$\frac{1}{2}+\frac{1}{6}\cos\kappa\sin 2\nu$	$\kappa=2n\pi,\nu=\frac{\pi}{4}+m\pi$
type4c	Alice(sender) $\rightarrow$ Bob $\leftrightarrow$ Cindy	W	$\frac{3}{4}$	
	Bob(sender) $\rightarrow$ Alice(co-sender) $\rightarrow$ Cindy(receiver)	GHZ	$\frac{1}{2}+\frac{1}{6}\cos\kappa\sin 2\nu$	$\kappa=2n\pi,\nu=\frac{\pi}{4}+m\pi$
	Bob(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Alice(receiver)	W	$\frac{3}{4}$	
type5	Alice(sender) $\rightarrow$ Bob $\leftrightarrow$ Cindy	W	$\frac{2}{3}$	
	Bob(sender) $\rightarrow$ Alice(co-sender) $\rightarrow$ Cindy(receiver)	GHZ	$\frac{8}{15}+\frac{2}{15}\cos 2\nu+\frac{2}{15}\cos\kappa\sin 2\nu$	$\kappa=2n\pi,\nu=\frac{\pi}{8}+m\pi$
	Bob(sender) $\rightarrow$ Cindy(co-sender) $\rightarrow$ Alice(receiver)	W	$\frac{2}{3}$	

TABLE VI: table for three party quantum teleportation

five three-qubit states to two classes, W and GHZ, based on protocols.

GHZ-type	W-type
Type2b	Type3, Type4, and Type5

III. CONCLUDING REMARK

In this report, we provided all schemes of quantum teleportation in three parties. Referring Acin et al.'s classification, which is based on the Schmidt decompostion with the set of strongly asymmetric basis, we considered all cases

of quantum teleportation in three parties. We obtained the best fidelity condition for each case. We also assigned the the roles(sender, co-sender, and receiver) of the parties generically.

Acknowledgement

Y. Kwon is supported by the Fund of Hanyang University.

[1] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A.Peres, and W. K. Wootters, "Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels", Phys. Rev. Lett. 70, 1895(1993).

[2] A. Karlsson and M. Bourennane, "Quantum teleportation using three-particle entanglement", Phys. Rev. A 58, 4393(1998).

[3] Y. Yeo, 'Quantum teleportation using three-particle entanglement', LANL Report No. quant-ph/0302030

[4] A. Acin, D. Brub, M. Lewenstein and A. Sanpera, "Clssification of mixed three- qubit states", Phys. Rev. Lett, 87, 040401(2001)